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EFFECT OF HALL CURRENT IN THERMOELASTIC MATERIALS WITH DOUBLE POROSITY STRUCTURE

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The present investigation is concerned with one dimensional problem in a homogeneous, isotropic thermoelastic medium with double porosity structure in the presence of Hall currents subjected to thermomechanical sources. A state space approach has been applied to investigate the problem. As an application of the approach, normal force and thermal source have been taken to illustrate the utility of the approach. The expressions for the components of normal stress, equilibrated stress and the temperature change are obtained in the frequency domain and computed numerically. A numerical simulation is prepared for these quantities. The effect of the Hartmann number is depicted graphically on the resulting quantities for a specific model. Some particular cases of interest are also deduced from the present investigation.

Key words: Hall current, double porosity, thermoelasticity, state space approach, thermomechanical sources.

1. Introduction

Porous media theories play an important role in many branches of engineering including the materials science, petroleum industry, chemical engineering, biomechanics and other fields of engineering. The representation of a fluid saturated porous medium as a single phase material has been virtually discarded. The material with pore spaces such as concrete can be treated easily because all concrete ingredients have the same motion if the concrete body is deformed. However, the situation is more complicated if the pores are filled with liquid and in that case the solid and liquid phases have different motions. Due to these different motions, different material properties and the complicated geometry of pore structures, the mechanical behavior of a fluid saturated porous thermoelastic medium becomes very difficult. So researchers have tried to overcome this difficulty and we can find many studies on porous media in the literature. A brief historical background of these theories is given by de Boer [1, 2].

The double porosity model represents a new possibility for the study of important problems concerning the civil engineering. It is well-known that, under super- saturation conditions due to water of other fluid effects, the so called neutral pressures generate unbearable stress states on the solid matrix and on the fracture faces, with severe (sometimes disastrous) instability effects like landslides, rock fall or soil fluidization (a typical phenomenon connected with propagation of seismic waves). In such a context, it seems possible, acting suitably on the boundary pressure state, to regulate the internal pressures in order to deactivate the noxious effects related to neutral pressures.

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Wilson and Aifanits [3] presented the theory of consolidation with the double porosity. Khaled *et al.* [4] employed a finite element method to consider the numerical solutions of the differential equation of the theory of consolidation with double porosity developed by Aifantis [3]. Wilson and Aifantis [5] discussed the propagation of acoustics waves in a fluid saturated porous medium. Various authors discussed different problems in double porous media [6]-[14]. Svanadze [15]-[19] investigated some problems on elastic solids, viscoelastic solids and thermoelastic solids with double porosity. Scarpetta *et al.* [20, 21] proved the uniqueness theorems in the theory of thermoelasticity for solids with double porosity and also obtained the fundamental solutions in the theory of thermoelasticity for solids with double porosity.

In recent years the state space description of linear systems has been used extensively in various areas of engineering, such as the analysis of control systems. The state space approach offers an attractive way to avoid the difficulties of the traditional linear model approach. The state –space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations. To abstract away from the number of inputs, outputs and states, the variables are expressed as vectors. If the dynamical system is linear and time invariant, the differential and algebraic equations may be written in a matrix form. The state-space representation provides a convenient and compact way to model and analyze systems with multiple inputs and outputs.

Bahar and Hetnarski [22]-[26] investigated a good number of problems in thermoelasticity by using the state space approach. Also Ezzat *et al.* [27], Maghraby *et al.*[28], Youssef and Al-Lehaibi [29], Othman [30], Elisbai and Youseff [31] and Sherief and El-Sayed [32] investigated different types of problems in different media by using the state space approach.

The foundations of magnetoelasticity were presented by Knopoff [33] and Chadwick [34] and developed by Kaliski and Petykiewicz [35]. Attention is paid to the interaction between the magnetic field and strain field in a thermoelastic solid due to its many applications in the fields of geophysics, plasma physics and related topics.

When the magnetic field is very strong, the conductivity will be a tensor and the effect of Hall current cannot be neglected. The conductivity normal to the magnetic field is reduced due to the free spiraling of electrons and ions about the magnetic lines of force before suffering collisions and a current is induced in a direction normal to both the electric and magnetic fields. This phenomenon is called the Hall effect. Authors such as Sarkar and Lahiri [36], Salem [37], Zakaria [38]-[40], Attia [41] have considered the effect of Hall currents for two dimensional problems in micropolar thermoelasticity.

In the present paper, we formulate the state space approach to the boundary value problem for a thermoelastic material with double porosity structure in the presence of Hall current subjected to thermomechanical sources. The expressions for normal stress, equilibrated stresses and temperature distribution are obtained in closed form, computed numerically and represented graphically for normal force and thermal source.

2 Basic equations

Following Iesan and Quintanilla [42], the field equations and the constitutive relations for a homogeneous thermoelastic material with double porosity structure, when the Hall current is taken into account, can be written as:
Eequation of motion

$$\mu\Delta u_i + (\lambda+\mu)u_{j,ji} + b\varphi_{,i} + d\psi_{,i} - \beta T_{,i} + F_i = \rho\ddot{u}_i \ , \tag{2.1}$$

equilibrated stress equations of motion

$$\alpha\Delta\varphi + b_1\Delta\psi - bu_{r,r} - \alpha_1\varphi - \alpha_3\psi + \gamma_1 T = \kappa_1\ddot{\varphi}, \tag{2.2}$$

$$b_1\Delta\varphi + \gamma\Delta\psi - du_{r,r} - \alpha_3\varphi - \alpha_2\psi + \gamma_2 T = \kappa_2\ddot{\psi} \ , \tag{2.3}$$

equation of heat conduction

$$\beta T_0 \dot{e}_{ii} + \gamma_1 T_0 \dot{\phi} + \gamma_2 T_0 \dot{\psi} + \rho C^* \dot{T} = K \nabla^2 T , \tag{2.4}$$

constitutive relations

$$t_{ij} = \lambda e_{rr} \delta_{ij} + 2\mu e_{ij} + b \delta_{ij} \varphi + d \delta_{ij} \psi - \beta \delta_{ij} T , \tag{2.5}$$

$$\sigma_i = \alpha \varphi_{,i} + b_1 \psi_{,i} , \tag{2.6}$$

$$\zeta_i = b_1 \varphi_{,i} + \gamma \psi_{,i} \tag{2.7}$$

where $F_i = \mu_0 \varepsilon_{ijr} J_j H_r$ is the Lorentz force.

The generalized Ohm's law including Hall current is

$$J_i = \sigma_0 \left(E_i + \mu_0 \varepsilon_{ijr} u_{j,t} H_r - \frac{\mu_0}{e n_e} \varepsilon_{ijr} J_j H_r \right) \tag{2.8}$$

where $\sigma_0 \left(= n_e e^2 t_e / m_e \right)$ is the electrical conductivity; μ_0 is the magnetic permeability; e is the charge of an electron; n_e is the number density of electrons; t_e is the electron collision time; m_e is the electron mass; E_i is the intensity tensor of the electric field; λ and μ are Lame's constants; ρ is the mass density; $\beta = (3\lambda + 2\mu)\alpha_t$; α_t is the coefficient of linear thermal expansion; C^* is the specific heat at constant strain; u_i is the displacement components; t_{ij} is the stress tensor; ε_{ijr} is the permutation symbol; μ_0 is the magnetic permeability; J_r is the conduction current density; κ_1 and κ_2 are coefficients of equilibrated inertia; ν_1 is the volume fraction field corresponding to pores and ν_2 is the volume fraction field corresponding to fissures; φ and ψ are the volume fraction fields corresponding to ν_1 and ν_2, respectively; σ_i is the equilibrated stress corresponding to ν_1; ζ_i is the equilibrated stress corresponding to ν_2, K is the coefficient of thermal conductivity and $b, d, b_1, \gamma, \gamma_1, \gamma_2$ are constitutive coefficients; δ_{ij} is the Kronecker's delta; T is the temperature change measured form the absolute temperature $T_0 (T_0 \neq 0)$, a superposed dot represents differentiation with respect to time variable t.

$$\nabla = \hat{i} \frac{\partial}{\partial x_1} + \hat{j} \frac{\partial}{\partial x_2} + \hat{k} \frac{\partial}{\partial x_3} , \qquad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} ,$$

are the gradient and Laplacian operators, respectively.

3. Formulation and solution of the problem

We consider a homogeneous, isotropic, perfectly conducting thermoelastic solid with double porosity occupying the region $0 \leq x < \infty$. For a one dimensional problem, we take $u_1(x_1,t), \varphi(x_1,t), \psi(x_1,t), T(x_1,t)$. A uniform very strong magnetic field of strength H_0 is assumed to be applied in the positive y –direction and we also assume that $E = 0$. Under these assumptions, the generalized Ohm's law gives $J_1 = J_2 = 0$ everywhere in the medium.

The current density components J_3 is given by

$$J_3 = \frac{\sigma_0 \mu_0 H_0}{1+m^2}\left(\frac{\partial u_1}{\partial t}\right) \tag{3.1}$$

where $m = \omega_e t_e$ is the Hall parameter and $\omega_e = e\mu_0 H_0 / m_e$ is the electron frequency.

Let us introduce the following non-dimensional variables

$$x_1' = \frac{\omega_1}{c_1}x_1, \quad u_1' = \frac{\omega_1}{c_1}u_1, \quad t_{ij}' = \frac{t_{ij}}{\beta T_0}, \quad M = \frac{\sigma_0 \mu_0^2 H_0^2}{\rho\omega}, \quad t' = \omega_1 t,$$
$$\varphi' = \frac{k_1 \omega_1{}^2}{\alpha_1}\varphi, \quad \psi' = \frac{k_1\omega_1{}^2}{\alpha_1}, \quad T' = \frac{T}{T_0}, \quad \sigma_i' = \left(\frac{c_1}{\alpha\omega_1}\right)\sigma_i, \quad \zeta_i' = \left(\frac{c_1}{\alpha\omega_1}\right)\zeta_i \tag{3.2}$$

where $c_1^2 = \dfrac{\lambda+2\mu}{\rho}, \omega_1 = \dfrac{\rho C^* c_1^2}{K}$ and M is the Hartmann number or magnetic parameter.

Making use of dimensionless quantities given in Eqs (3.2) in Eqs (2.1)-(2.4), (dropping primes for convenience), and assuming the time harmonic solution of the resulting equations, we obtain after some simplifications

$$\overline{u}_{,11} = N_1\overline{u} + N_2\overline{\phi}_{,1} + N_3\overline{\psi}_{,1} + N_4\overline{T}_{,1}, \tag{3.3}$$

$$\overline{\phi}_{,11} = N_5\overline{u}_{,1} + N_6\overline{\phi} + N_7\overline{\psi} + N_8\overline{T}, \tag{3.4}$$

$$\overline{\psi}_{,11} = N_9\overline{u}_{,1} + N_{10}\overline{\phi} + N_{11}\overline{\psi} + N_{12}\overline{T}, \tag{3.5}$$

$$\overline{T}_{,11} = N_{13}\overline{u}_{,1} + N_{14}\overline{\phi} + N_{15}\overline{\psi} + N_{16}\overline{T} \tag{3.6}$$

where

$$N_1 = -i\omega\left(\frac{M}{1+m^2}\right) - \omega^2, \quad N_2 = -\delta_1, \quad N_3 = -\delta_2, \quad N_4 = \delta_3, \quad M_1 = \frac{-\delta_5}{\delta_4}, \quad M_2 = \frac{\delta_6}{\delta_4},$$

$$M_3 = \frac{\delta_7 - \omega^2}{\delta_4}, \quad M_4 = \frac{\delta_8}{\delta_4}, \quad M_5 = \frac{-\delta_9}{\delta_4}, \quad M_6 = \frac{-\delta_{10}}{\delta_{11}}, \quad M_7 = \frac{\delta_{12}}{\delta_{11}}, \quad M_8 = \frac{\delta_{13}}{\delta_{11}},$$

$$M_9 = \frac{\delta_{14} - \omega^2}{\delta_{11}}, \quad M_{10} = \frac{-\delta_{15}}{\delta_{11}}, \quad M_{11} = \frac{\delta_{17}}{\delta_{20}}, \quad M_{12} = \frac{\delta_{18}}{\delta_{20}}, \quad M_{13} = \frac{\delta_{19}}{\delta_{20}}, \quad M_{14} = \frac{1}{\delta_{20}},$$

$$\delta_{20} = \frac{\delta_{16}}{-i\omega}, \quad N_{13} = \delta_{16}, \quad N_{14} = \delta_{17}, \quad N_{15} = \delta_{18}, \quad N_{16} = 1, \tag{3.7}$$

$$M_{15} = 1 - M_1M_6, \quad N_5 = \frac{M_1M_7 + M_2}{M_{15}}, \quad N_6 = \frac{M_1M_8 + M_3}{M_{15}}, \quad N_7 = \frac{M_1M_9 + M_4}{M_{15}},,$$

$$N_8 = \frac{M_1 M_{10} + M_5}{M_{15}}, \quad N_9 = M_6 N_5 + M_7, \quad N_{10} = M_6 N_6 + M_8, \quad N_{11} = M_6 N_7 + M_9,$$

$$N_{12} = M_6 N_8 + M_{10}, \quad \delta_1 = \frac{b\alpha_1}{\rho C_1^2 k_1 \omega_1{}^2}, \quad \delta_2 = \frac{d\alpha_1}{\rho C_1^2 k_1 \omega_1{}^2}, \quad \delta_3 = \frac{\beta T_0}{\rho C_1^2}, \quad \delta_4 = \frac{\alpha}{C_1^2 k_1},$$

$$\delta_5 = \frac{b_1}{C_1^2 k_1}, \quad \delta_6 = \frac{b}{\alpha_1}, \quad \delta_7 = \frac{\alpha_1}{k_1 \omega_1{}^2}, \quad \delta_8 = \frac{\alpha_3}{k_1 \omega_1{}^2}, \quad \delta_9 = \frac{\gamma_1 T_0}{\alpha_1}, \quad \delta_{10} = \frac{b_1}{C_1^2 k_2},$$

$$\delta_{11} = \frac{\gamma}{C_1^2 k_2}, \quad \delta_{12} = \frac{dk_1}{\alpha_1 k_2}, \quad \delta_{13} = \frac{\alpha_3}{k_2 \omega_1{}^2}, \quad \delta_{14} = \frac{\alpha_2}{k_2 \omega_1{}^2},$$

$$\delta_{15} = \frac{\gamma_2 T_0 k_1}{\alpha_1 k_2}, \quad \delta_{16} = \frac{\beta C_1^2}{K\omega_1}, \quad \delta_{17} = \frac{\gamma_1 \alpha_1 C_1^2}{K k_1 \omega_1{}^3}, \quad \delta_{18} = \frac{\gamma_2 \alpha_1 C_1^2}{K k_1 \omega_1{}^3}.$$

4. State -space formulation

Choosing as state variables the displacement $\bar{u}$, volume fraction $\bar{\varphi}$ and $\bar{\psi}$, temperature change $\bar{T}$ in the x- direction, then the equations can be written in the matrix form as

$$\frac{dV(x,\omega)}{dx} = A(\omega)V(x,\omega) \tag{4.1}$$

and the values of $A(\omega), V(x,\omega)$ are given in Appendix I.

The formal solution of the system (4.1) can be written in the form

$$V(x,\omega) = \exp\left[A(\omega)x\right]V(0,\omega) \ . \tag{4.2}$$

The value of $V(0,\omega)$ is given in Appendix I.

We shall use the well-known Cayley-Hamilton theorem to find the form of the matrix $\exp[A(\omega)x]$. The characteristics equation of the matrix $A(\omega)$ can be written as

$$\lambda^8 + D_1\lambda^6 + D_2\lambda^4 + D_3\lambda^2 + D_4 = 0 \tag{4.3}$$

where

$$D_1 = -N_1 - N_6 - N_{11} - N_{16} - N_2 N_5 - N_3 N_9 - N_4 N_{13},$$

$$\begin{aligned} D_2 = {} & N_1 N_6 + N_1 N_{11} + N_1 N_{16} + N_6 N_{11} + N_6 N_{16} - N_7 N_{10} - N_8 N_{14} + N_{11} N_{16} + \\ & - N_{12} N_{15} - N_2 N_7 N_9 + N_3 N_6 N_9 + N_2 N_5 N_{11} + N_2 N_5 N_{16} - N_3 N_5 N_{10} + N_3 N_9 N_{16} + \\ & - N_2 N_8 N_{13} - N_4 N_5 N_{14} + N_4 N_6 N_{13} - N_4 N_9 N_{15} - N_3 N_{12} N_{13} + N_4 N_{11} N_{13}, \end{aligned}$$

$$D_3 = -N_6 N_{11} N_{16} + N_7 N_{10} N_{16} - N_1 N_6 N_{11} + N_1 N_7 N_{10} - N_1 N_6 N_{16} + N_1 N_8 N_{14} - N_1 N_{11} N_{16} + $$
$$+ N_1 N_{12} N_{15} + N_6 N_{12} N_{15} - N_7 N_{12} N_{14} - N_8 N_{10} N_{15} + N_8 N_{11} N_{14} + N_2 N_7 N_9 N_{16} - N_3 N_6 N_9 N_{16} + $$
$$- N_2 N_8 N_9 N_{15} + N_3 N_8 N_9 N_{14} + N_4 N_6 N_9 N_{15} - N_4 N_7 N_9 N_{14} - N_2 N_5 N_{11} N_{16} + N_3 N_5 N_{10} N_{16} + $$
$$+ N_2 N_5 N_{12} N_{15} - N_2 N_7 N_{12} N_{13} + N_2 N_8 N_{11} N_{13} - N_3 N_5 N_{12} N_{14} + N_3 N_6 N_{12} N_{13} - N_3 N_8 N_{10} N_{13} + $$
$$- N_4 N_5 N_{10} N_{15} + N_4 N_5 N_{11} N_{14} - N_4 N_6 N_{11} N_{13} + N_4 N_7 N_{10} N_{13},$$
$$D_4 = N_1 N_6 (N_{11} N_{16} - N_{12} N_{15}) + N_1 N_7 (N_{12} N_{14} - N_{10} N_{16}) + N_1 N_8 (N_{10} N_{15} - N_{11} N_{14}) \quad (4.4)$$

Equation (4.3) is biquadrate in λ^2, yields four roots: $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Now the Taylor series expansion for the matrix exponential in Eq.(4.2) is given by

$$\exp\left[A(\omega)x\right] = \sum_{n=0}^{\infty} \left\{ \frac{\left[A(\omega)x\right]^n}{n!} \right\}. \quad (4.5)$$

Using the Cayley-Hamilton theorem, this infinite series can be truncated as

$$\exp\left[A(\omega)x\right] = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 \quad (4.6)$$

where a_0, a_1, a_2, a_3 are parameters depending on x and ω.

According to the Cayley-Hamilton theorem the characteristic roots $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ of the matrix A must satisfy Eq.(4.6). Therefore, we get

$$\exp\left[-\lambda_1 x\right] = a_0 I - a_1 \lambda_1 + a_2 \lambda_1^2 - a_3 \lambda_1^3,$$

$$\exp\left[-\lambda_2 x\right] = a_0 I - a_1 \lambda_2 + a_2 \lambda_2^2 - a_3 \lambda_2^3, \quad (4.7)$$

$$\exp\left[-\lambda_3 x\right] = a_0 I - a_1 \lambda_3 + a_2 \lambda_3^2 - a_3 \lambda_3^3,$$

$$\exp\left[-\lambda_4 x\right] = a_0 I - a_1 \lambda_4 + a_2 \lambda_4^2 - a_3 \lambda_4^3.$$

Solving the above system of equations, we obtain the value of parameters a_0, a_1, a_2, a_3 and these values are given in Appendix I.

Therefore, we have

$$\exp\left[A(\omega)x\right] = L(x, \omega) \quad (4.8)$$

where $L(x, w)$ is a 8×8 matrix with the components

$$l_{11} = a_0 + a_2 N_1, \quad l_{12} = a_3 R_1, \quad l_{13} = a_3 R_2, \quad l_{14} = a_3 R_3, \quad l_{21} = a_3 R_5, \quad l_{22} = a_0 + a_2 N_6,$$

$$l_{23} = a_2 N_7, \; l_{24} = a_2 N_8, \; l_{31} = a_3 R_9, \; l_{32} = a_2 N_{10}, \; l_{33} = a_0 + a_2 N_{11}, \; l_{34} = a_2 N_{12}, \; l_{41} = a_3 R_{13},$$

$$l_{42} = a_2 N_{14}, \quad l_{43} = a_2 N_{15}, \quad l_{44} = a_0 + a_2 N_{16}, \quad R_1 = N_2 N_6 + N_3 N_{10} + N_4 N_{14}, \tag{4.9}$$

$$R_2 = N_2 N_7 + N_3 N_{11} + N_4 N_{15}, \qquad R_3 = N_2 N_8 + N_3 N_{12} + N_4 N_{16},$$

$$R_5 = N_1 N_5, \quad R_9 = N_1 N_9, \quad R_{13} = N_1 N_{13}.$$

Rewriting Eq.(4.2) with the aid of Eq.(4.8) yields

$$V(x,\omega) = L(x,\omega) V(0,\omega). \tag{4.10}$$

Therefore, we obtain

$$\begin{bmatrix} \bar{u} \\ \bar{\varphi} \\ \bar{\psi} \\ \bar{T} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ l_{21} & l_{22} & l_{23} & l_{24} \\ l_{31} & l_{32} & l_{33} & l_{34} \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \tag{4.11}$$

5. Boundary conditions

A homogeneous isotropic thermoelastic solid with double porosity structure occupying the region $0 \le x < \infty$ is considered. The bounding plane $x = 0$ is subjected to normal force and thermal source. Mathematically these can be written as

(i) $$t_{11} = -F_1 \exp[-i\omega t], \tag{5.1}$$

(ii) $$\sigma_1 = -F_1 \exp[-i\omega t], \tag{5.2}$$

(iii) $$\zeta_1 = -F_1 \exp[-i\omega t], \tag{5.3}$$

(iv) $$T = F_2 \exp[-i\omega t] \tag{5.4}$$

where F_1 and F_2 are the magnitude of the force and constant temperature applied on the boundary, respectively.

Substituting the values of $u, \varphi, \psi, T, t_{11}, \sigma_1$ and ζ_1 from Eqs (2.1), (2.2), (2.3), (4.10) in to Eqs (5.1)-(5.4) and with the aid of Eqs (3.1) and (3.7), after some lengthy calculations, we obtain the normal stress, equilibrated stresses and temperature change as

$$t_{11} = \left(S_1 \frac{\Gamma_1}{\Gamma} + S_2 \frac{\Gamma_2}{\Gamma} + S_3 \frac{\Gamma_3}{\Gamma} + S_4 \frac{\Gamma_4}{\Gamma} \right) e^{-i\omega t}, \tag{5.5}$$

$$\sigma_1 = \left(S_5 \frac{\Gamma_1}{\Gamma} + S_6 \frac{\Gamma_2}{\Gamma} + S_7 \frac{\Gamma_3}{\Gamma} + S_8 \frac{\Gamma_4}{\Gamma} \right) e^{-i\omega t}, \tag{5.6}$$

$$\zeta_1 = \left(S_9 \frac{\Gamma_1}{\Gamma} + S_{10} \frac{\Gamma_2}{\Gamma} + S_{11} \frac{\Gamma_3}{\Gamma} + S_{12} \frac{\Gamma_4}{\Gamma} \right) e^{-i\omega t}, \tag{5.7}$$

$$T=\left(l_{41}\frac{\Gamma_1}{\Gamma}+l_{42}\frac{\Gamma_2}{\Gamma}+l_{43}\frac{\Gamma_3}{\Gamma}+l_{44}\frac{\Gamma_4}{\Gamma}\right)e^{-i\omega t}. \tag{5.8}$$

The values of $S_1, S_2, \ldots\ldots\ldots\ldots, S_{12}$ are given in Appendix II.

6. Particular cases

Case 6.1: If $F_2 = 0$ in Eqs (5.5)-(5.8), we obtain the corresponding expressions for normal force.

Case 6.2: If $F_1 = 0$ in Eqs (5.5)-(5.8), we get the corresponding expressions for thermal source.

7. Numerical results and discussion

The material chosen for the purpose of numerical computation is copper, whose physical data is given by Sherief and Saleh [43] as

$$\lambda = 7.76\times10^{10}\,Nm^{-2},\quad c^* = 3.831\times10^3\,m^2s^{-2}K^{-1},\quad \mu = 3.86\times10^{10}\,Nm^{-2},$$

$$k = 3.86\times10^3\,Ns^{-1}K^{-1},\quad \omega = 1\times10^{11}\,\mathrm{s}^{-1},\quad T_0 = 0.293\times10^3\,K,$$

$$\alpha_t = 1.78\times10^{-5}\,K^{-1},\quad t = 0.1s,\quad \rho = 8.954\times10^3\,Kgm^{-3}.$$

Following Khalili [44], the double porous parameters are taken as

$$\alpha_2 = 2.4\times10^{10}\,Nm^{-2},\quad \alpha_3 = 2.5\times10^{10}\,Nm^{-2},\quad \alpha = 1.3\times10^{-5}\,N,\quad \gamma = 1.1\times10^{-5}\,N,$$

$$\gamma_1 = 0.16\times10^5\,Nm^{-2},\quad b_1 = 0.12\times10^{-5}\,N,\quad d = 0.1\times10^{10}\,Nm^{-2},$$

$$\gamma_2 = 0.219\times10^5\,Nm^{-2},\quad k_1 = 0.1456\times10^{-12}\,Nm^{-2}s^2,\quad b = 0.9\times10^{10}\,Nm^{-2},$$

$$\alpha_1 = 2.3\times10^{10}\,Nm^{-2},\quad k_2 = 0.1546\times10^{-12}\,Nm^{-2}s^2.$$

Following Zakaria [40], the electric constants are taken as

$$\sigma_0 = 9.36\times10^5\,Col^2/Cl.cm.s,$$

$$H_0 = 10^8\,Col/cm.\ s.$$

The software MATLAB has been used to determine the values of normal stress, equilibrated stresses and temperature change. Figures 1-4 and Figs 5-8 depict the variations of normal stress, equilibrated stresses and temperature distribution with the Hartmann number (M) with respect to distance x for normal force and thermal source, respectively. In all the figures, the solid line corresponds to the value of $M = 0$, small dashed line corresponds to the value of $M = 1$ and big dashed line corresponds to the value of $M = 1.5$.

Normal Force

Figure 1 shows the variation of normal stress t_{11} with respect to distance *x*. The variation is similar for all values of the Hartmann number. It is noticed that with the increase in the value of M, the value of normal stress also increases.

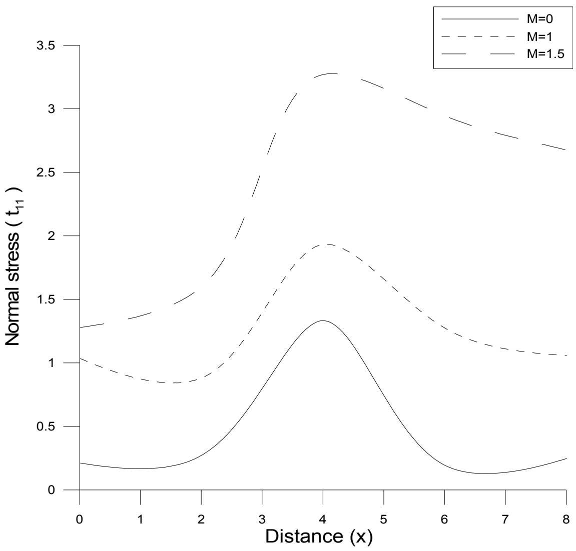


Fig.1. Variation of normal stress t_{11} w.r.t. *x*.

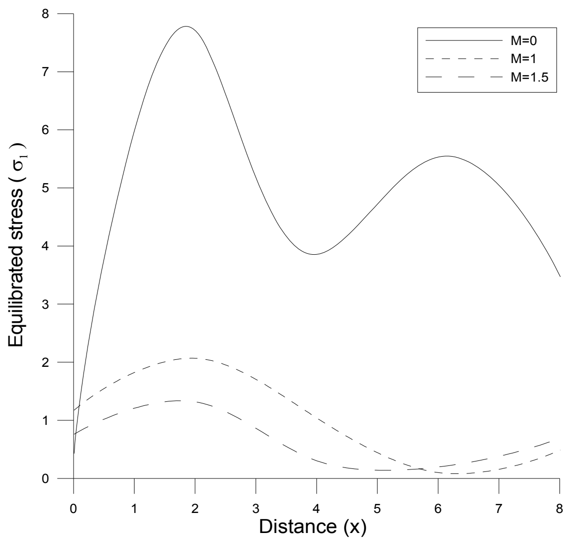


Fig.2. Variation of equilibrated stress σ_1 w.r.t *x*.

Figures 2 and 3 depict the variations of equilibrated stresses σ_1 and τ_1 with respect to distance *x*, respectively. For $M = 0$, the value of σ_1 and τ_1 increases for $0 < x < 2$, again decreases for $2 \le x < 4$ and then again increases for $4 \le x < 6$ and further decreases away from the source. For $M = 1$ and 1.5, a similar behavior is noticed near the application of the source whereas on opposite behavior is noticed away from the source.

Figure 4 represents the variation of temperature change *T* with respect to distance *x*. It is found that the behavior is similar for $M = 0$ and 1 while it becomes reverse for $M = 1.5$.

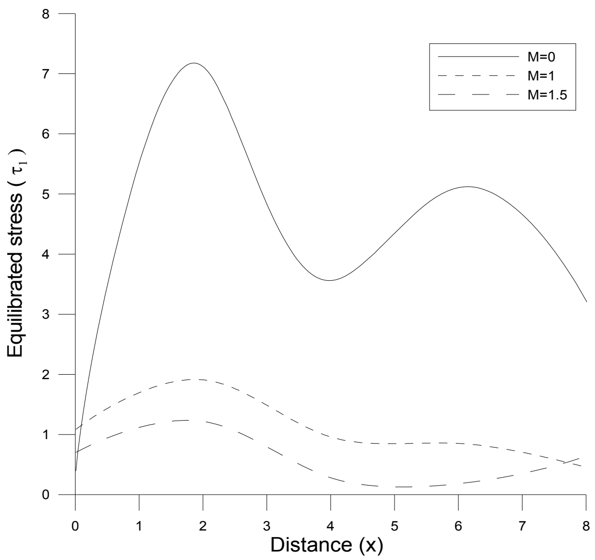


Fig.3. Variation of equilibrated stress ζ_1 w.r.t. *x*.

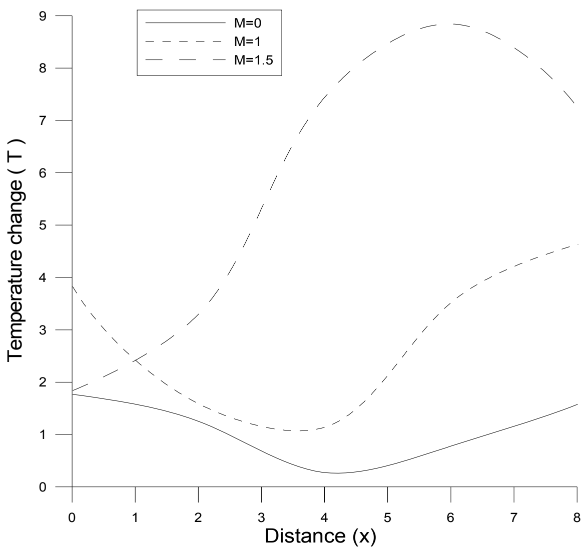
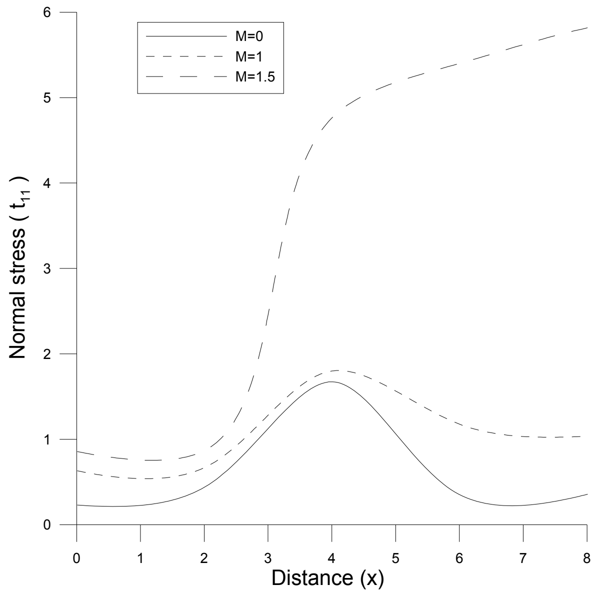


Fig.4. Variation of temperature change *T* w.r.t *x*.

Thermal Source

Figure 5 depicts the variation of normal stress t_{11} with respect to distance x. The variation is similar for all the three cases under consideration $(M = 0, 1, 1.5)$. It is noticed that with the increase in the value of M, the value of normal stress also increases.



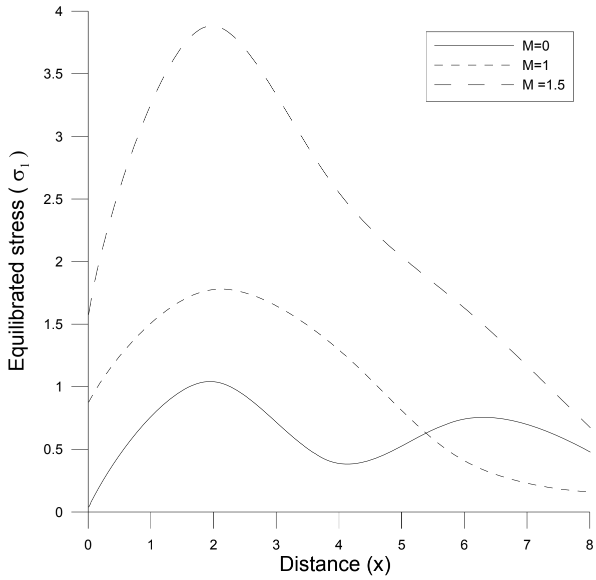
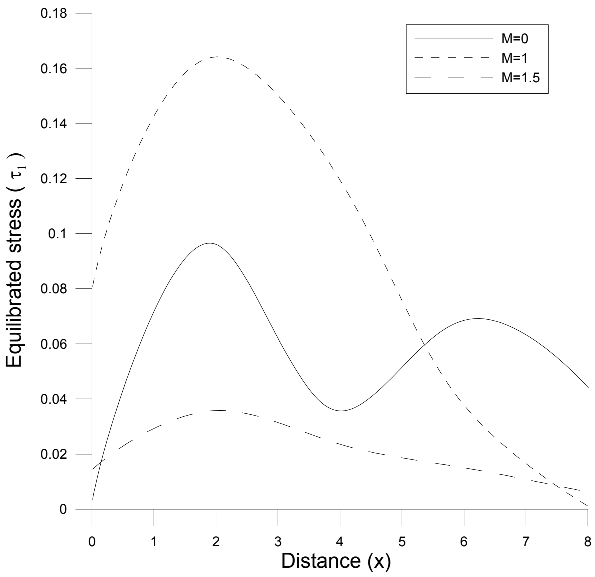


Fig.5. Variation of normal stress t_{11} w.r.t. x.

Fig.6. Variation of equilibrated stress σ_1 w.r.t x.

Figures 6 and 7 show the variation of equilibrated stresses σ_1 and τ_1 with respect to distance x, respectively. The variation is of oscillatory nature for $M = 0$ while the same behavior is noticed for $M = 1$ and 1.5 , i.e., monotonically increasing and decreasing.



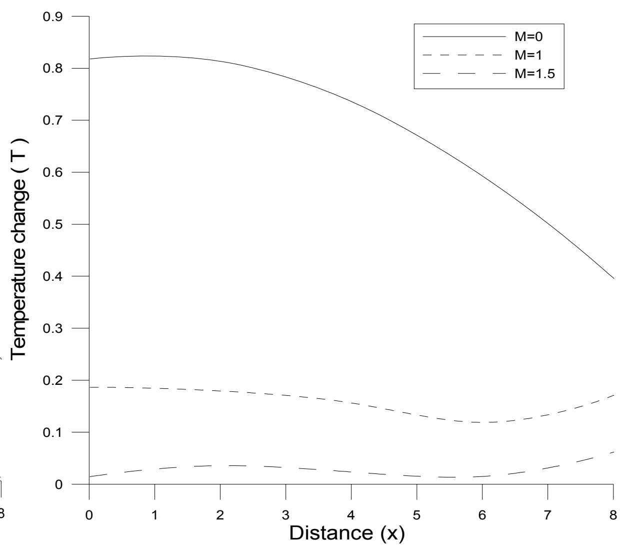


Fig.7. Variation of equilibrated stress ζ_1 w.r.t. x.

Fig.8. Variation of temperature change T w.r.t x.

Figure 8 represents the variation of temperature change T with respect to distance x. It is noticed that with the increase in the value of M, the value of temperature change decreases.

8 Conclusion

The behaviour of normal stress, equilibrated stresses and temperature distribution in an isotropic homogeneous thermoelastic material with double porosity structure under the effect of Hall currents has been investigated for the normal force and thermal source by using the state space approach. It is observed that with the increase in the value of the Hartmann number, normal stress also increases. The behavior of equilibrated stresses is oscillatory in nature for $M = 0$ where for $M = 1$ and 1.5, the behavior is same near the application of the source while a reverse behavior is observed away from the source. For normal force, the behavior of temperature change is similar for $M = 0$ and 1, whereas an opposite behavior is observed for $M = 1.5$ while in the case of thermal source, the value of temperature changes decreases with the increase in value of the Hartmann number.

Nomenclature

$b, d, b_1, \gamma, \gamma_1, \gamma_2$ - constitutive coefficients
C^* – specific heat at constant strain
E_i – intensity tensor of the electric field
e – charge of an electron
J_r – conduction current density
K – coefficient of thermal conductivity
m_e – electron mass
n_e – number density of electrons
$T = T^* - T_0$ – small temperature increment
t_e – electron collision time
t_{ij} – stress tensor
u_i – displacement components
α_t – coefficient of linear thermal expansion
δ_{ij} – Kronecker's delta
ε_{ijr} – permutation symbol
ζ_i – equilibrated stress corresponding to ν_2
κ_1, κ_2 – coefficients of equilibrated inertia
λ, μ – Lame's constants
μ_0 – magnetic permeability
ν_1 – volume fraction field corresponding to pores
ν_2 – volume fraction field corresponding to fissures
ρ – mass density
σ_0 – electrical conductivity
σ_i – equilibrated stress corresponding to ν_1
φ – volume fraction field corresponding to ν_1
ψ – volume fraction field corresponding to ν_2

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APPENDIX I

$$A(x,w)=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ N_1 & 0 & 0 & 0 & 0 & N_2 & N_3 & N_4 \\ 0 & N_6 & N_7 & N_8 & N_5 & 0 & 0 & 0 \\ 0 & N_{10} & N_{11} & N_{12} & N_9 & 0 & 0 & 0 \\ 0 & N_{14} & N_{15} & N_{16} & N_{13} & 0 & 0 & 0 \end{bmatrix},$$

$$V(x,w)=\begin{bmatrix} \bar{u}(x,w) \\ \bar{\varphi}(x,w) \\ \bar{\psi}(x,w) \\ \bar{T}(x,w) \\ (\bar{u}(x,w))_{,1} \\ (\bar{\varphi}(x,w))_{,1} \\ (\bar{\psi}(x,w))_{,1} \\ (\bar{T}(x,w))_{,1} \end{bmatrix}, \qquad V(0,w)=\begin{bmatrix} \bar{u}(0w) \\ \bar{\varphi}(0,w) \\ \bar{\psi}(0,w) \\ \bar{T}(0,w) \\ (\bar{u}(0,w))_{,1} \\ (\bar{\varphi}(0,w))_{,1} \\ (\bar{\psi}(0,w))_{,1} \\ (\bar{T}(0,w))_{,1} \end{bmatrix},$$

$$a_0=e^{-\lambda_1 x}\left[1-\frac{\lambda_1\lambda_2(\lambda_3+\lambda_4)+\lambda_1\lambda_3\lambda_4}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}+\frac{\lambda_1^{2}(\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}+\right.$$
$$\left.-\frac{\lambda_1^{3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\right]-e^{-\lambda_2 x}\left[\frac{\lambda_1^{2}(\lambda_3+\lambda_4)+\lambda_1\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}+\right.$$
$$\left.-\frac{\lambda_1^{2}(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4))}+\frac{\lambda_1^{3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\right]+$$
$$-e^{-\lambda_3 x}\left[\frac{\lambda_1^{2}(\lambda_2+\lambda_4)+\lambda_1\lambda_2\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}-\frac{\lambda_1^{2}(\lambda_1+\lambda_2+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4))}+\right.$$
$$\left.+\frac{\lambda_1^{3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_{3s}-\lambda_4)}\right]-e^{-\lambda_4 x}\left[\frac{\lambda_1^{2}(\lambda_2+\lambda_3)+\lambda_1\lambda_2\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}+\right.$$
$$\left.-\frac{\lambda_1^{2}(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}+\frac{\lambda_1^{3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}\right],$$

$$a_1 = -e^{-\lambda_1 x}\left[\frac{\lambda_2(\lambda_3+\lambda_4)+\lambda_3\lambda_4}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\right]-e^{-\lambda_2 x}\left[\frac{\lambda_1(\lambda_3+\lambda_4)+\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\right]+$$
$$-e^{-\lambda_3 x}\left[\frac{\lambda_1(\lambda_2+\lambda_4)+\lambda_2\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}\right]-e^{-\lambda_4 x}\left[\frac{\lambda_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}\right],$$

$$a_2 = -e^{-\lambda_1 x}\left[\frac{(\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\right]-e^{-\lambda_2 x}\left[\frac{(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\right]+$$
$$-e^{-\lambda_3 x}\left[\frac{(\lambda_1+\lambda_2+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}\right]-e^{-\lambda_4 x}\left[\frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}\right],$$

$$a_3 = -e^{-\lambda_1 x}\left[\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}\right]-e^{-\lambda_2 x}\left[\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\right]+$$
$$-e^{-\lambda_3 x}\left[\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}\right]-e^{-\lambda_4 x}\left[\frac{1}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}\right].$$

APPENDIX II

$$Q_1 = P_1(Z_1+N_1Z_3)+P_2\left(a_3^0R_5\right)+P_3\left(a_3^0R_9\right)-a_3^0R_{13},$$

$$Q_2 = P_1R_1Z_4+P_2\left(a_0^0+a_2^0N_6\right)+P_3\left(a_0^0+a_2^0N_{11}\right)-a_2^0N_{14},$$

$$Q_3 = P_1R_2Z_4+P_2\left(a_2^0N_7\right)+P_3\left(a_0^0+a_2^0N_{11}\right)-a_2^0N_{15},$$

$$Q_4 = P_1R_3Z_4+P_2\left(a_2^0N_8\right)+P_3\left(a_2^0N_{12}\right)-\left(a_0^0+a_2^0N_{16}\right),$$

$$Q_5 = P_4R_5Z_4+P_5R_9Z_4, Q_6 = P_4\left(Z_1+N_6Z_3\right)+P_5N_{10}Z_3,$$

$$Q_7 = P_4N_7Z_3+P_5\left(Z_1+N_{11}Z_3\right), Q_8 = P_4N_8Z_3+P_5N_{12}Z_3,$$

$$Q_9 = P_5R_5Z_4+P_6R_9Z_4, Q_{10} = P_5\left(Z_1+N_6Z_3\right)+P_6N_{10}Z_3,$$

$$Q_{11} = P_5N_7Z_3+P_6\left(Z_1+N_{11}Z_3\right), Q_{12} = P_5N_8Z_3+P_6N_{12}Z_3,$$

$$Q_{13} = a_3^0R_{13}, Q_{14} = a_2^0N_{14}, Q_{15} = a_2^0N_{15}, Q_{16} = a_0^0+a_2^0N_{16}$$

where

$$P_1 = \frac{\lambda+2\mu}{\beta T_0},\quad P_2 = \frac{b\alpha_1}{k_1\omega^2\beta T_0},\quad P_3 = \frac{d\alpha_1}{k_1\omega^2\beta T_0},\quad P_4 = \frac{\alpha_1}{k_1\omega^2},\quad P_5 = \frac{b_1\alpha_1}{\alpha k_1\omega^2},\quad P_6 = \frac{\gamma\alpha_1}{\alpha k_1\omega^2},$$

$$Z_1 = -\lambda_1 D_{11} - \lambda_2 D_{12} - \lambda_3 D_{13} - \lambda_4 D_{14}, \qquad Z_2 = -\lambda_1 D_{21} - \lambda_2 D_{22} - \lambda_3 D_{23} - \lambda_4 D_{24},$$

$$Z_3 = -\lambda_1 D_{31} - \lambda_2 D_{32} - \lambda_3 D_{33} - \lambda_4 D_{34}, \qquad Z_4 = -\lambda_1 D_{41} - \lambda_2 D_{42} - \lambda_3 D_{43} - \lambda_4 D_{44},$$

$$Y_1 = -\lambda_1 D_{11} e^{-\lambda_1 x} - \lambda_2 D_{12} e^{-\lambda_2 x} - \lambda_3 D_{13} e^{-\lambda_3 x} - \lambda_4 D_{14} e^{-\lambda_4 x},$$

$$Y_2 = -\lambda_1 D_{11} e^{-\lambda_1 x} - \lambda_2 D_{12} e^{-\lambda_2 x} - \lambda_3 D_{13} e^{-\lambda_3 x} - \lambda_4 D_{14} e^{-\lambda_4 x},$$

$$Y_3 = -\lambda_1 D_{21} e^{-\lambda_1 x} - \lambda_2 D_{22} e^{-\lambda_2 x} - \lambda_3 D_{23} e^{-\lambda_3 x} - \lambda_4 D_{24} e^{-\lambda_4 x},$$

$$Y_4 = -\lambda_1 D_{31} e^{-\lambda_1 x} - \lambda_2 D_{32} e^{-\lambda_2 x} - \lambda_3 D_{33} e^{-\lambda_3 x} - \lambda_4 D_{34} e^{-\lambda_4 x},$$

$$S_1 = P_1(Y_1 + N_1 Y_3) + P_2(l_{21}) + P_3 l_{31} - l_{41}, \qquad S_2 = P_1(R_1 Y_4) + P_2(l_{22}) + P_3(l_{32}) - l_{42},$$

$$S_3 = P_1(R_2 Y_4) + P_2(l_{23}) + P_3(l_{33}) - l_{43}, \qquad S_4 = P_1 R_3 Y_4 + P_2(l_{24}) + P_3(l_{34}) - (l_{44}),$$

$$S_5 = P_4 R_5 Y_4 + P_5 R_9 Y_4, \qquad S_6 = P_4(Y_4 + N_6 Y_3) + P_5 N_{10} Y_3,$$

$$S_7 = P_4 N_7 Y_3 + P_5(Y_1 + N_{11} Y_3), \qquad S_8 = P_4 N_8 Y_3 + P_5 N_{12} Y_3$$

$$S_9 = P_5 R_5 Y_4 + P_6 R_9 Y_4, \qquad S_{10} = P_5(Y_1 + N_6 Y_3) + P_6 N_{10} Y_3,$$

$$S_{11} = P_5 N_7 Y_3 + P_6(Y_1 + N_{11} Y_3), \qquad S_{12} = P_5 N_8 Y_3 + P_6 N_{12} Y_3$$

$$D_{11} = 1 - \frac{\lambda_1\lambda_2(\lambda_3+\lambda_4) + \lambda_1\lambda_3\lambda_4}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_1^2(\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \\ - \frac{\lambda_1^3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)},$$

$$D_{12} = -\left[\frac{\lambda_1^2(\lambda_3+\lambda_4) + \lambda_1\lambda_3\lambda_4}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} - \frac{\lambda_1^2(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4))} + \right. \\ \left. + \frac{\lambda_1^3}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\right],$$

$$D_{13} = -\left[\frac{\lambda_1^2(\lambda_2+\lambda_4) + \lambda_1\lambda_2\lambda_4}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} - \frac{\lambda_1^2(\lambda_1+\lambda_2+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \right. \\ \left. + \frac{\lambda_1^3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}\right],$$

$$D_{14} = -\left[\frac{\lambda_1^2(\lambda_2+\lambda_3)+\lambda_1\lambda_2\lambda_3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} - \frac{\lambda_1^2(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} + \right.$$

$$\left. + \frac{\lambda_1^3}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}\right],$$

$$D_{31} = -\frac{(\lambda_2+\lambda_3+\lambda_4)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}, \qquad D_{32} = -\frac{(\lambda_1+\lambda_3+\lambda_4)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$

$$D_{33} = -\frac{(\lambda_1+\lambda_2+\lambda_4)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}, \qquad D_{34} = -\frac{(\lambda_1+\lambda_2+\lambda_3)}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)},$$

$$D_{41} = -\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}, \qquad D_{42} = -\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)},$$

$$D_{43} = -\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}, \qquad D_{44} = -\frac{1}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)},$$

$$\Gamma = \begin{vmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ Q_5 & Q_6 & Q_7 & Q_8 \\ Q_9 & Q_{10} & Q_{11} & Q_{12} \\ Q_{13} & Q_{14} & Q_{15} & Q_{16} \end{vmatrix}, \qquad \Gamma_1 = \begin{vmatrix} -F_1 & Q_2 & Q_3 & Q_4 \\ -F_1 & Q_6 & Q_7 & Q_8 \\ -F_1 & Q_{10} & Q_{11} & Q_{12} \\ F_2 & Q_{14} & Q_{15} & Q_{16} \end{vmatrix},$$

$$\Gamma_2 = \begin{vmatrix} Q_1 & -F_1 & Q_3 & Q_4 \\ Q_5 & -F_1 & Q_7 & Q_8 \\ Q_9 & -F_1 & Q_{11} & Q_{12} \\ Q_{13} & F_2 & Q_{15} & Q_{16} \end{vmatrix}, \qquad \Gamma_3 = \begin{vmatrix} Q_1 & Q_2 & -F_1 & Q_4 \\ Q_5 & Q_6 & -F_1 & Q_8 \\ Q_9 & Q_{10} & -F_1 & Q_{12} \\ Q_{13} & Q_{14} & F_2 & Q_{16} \end{vmatrix},$$

$$\Gamma_4 = \begin{vmatrix} Q_1 & Q_2 & Q_3 & -F_1 \\ Q_5 & Q_6 & Q_7 & -F_1 \\ Q_9 & Q_{10} & Q_{11} & -F_1 \\ Q_{13} & Q_{14} & Q_{15} & F_2 \end{vmatrix},$$

and $a_0^0 = a_0$, $a_2^0 = a_2$, $a_3^0 = a_3$, at $x = 0$, $A_1 = \frac{\Gamma_1}{\Gamma}$, $A_2 = \frac{\Gamma_2}{\Gamma}$, $A_3 = \frac{\Gamma_3}{\Gamma}$, $A_4 = \frac{\Gamma_4}{\Gamma}$.